Quantum operational measurement of amplitude and phase parameters for $SU(3)$ symmetry optical fields

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Abstract
We consider a new approach for describing a quantum optical Bose system with internal Gell-Mann symmetry by means of the $SU(3)$ symmetry polarization map in Hilbert space. The operational measurement of the density (or coherency) matrix elements for the three-mode optical field is discussed for the first time. We have introduced a set of operators that describe the quantum measurement procedure and the behaviour of fluctuations for the amplitude and phase characteristics of the three-level system. A novel twelve-port interferometer for making parallel measurements of the Gell-Mann parameters is proposed. The quantum properties of qutrit W-states in the measurement procedure are examined.

Keywords: quantum measurement, squeezing, tomography, polarization, entanglement, qutrit, $SU(3)$ symmetry

1. Introduction
At present, the properties of three-level optical and atomic systems (qutrit states) are evoking great interest in relation to modern problems of quantum information and communication [1–5]. In fact, implementation of qutrit states in quantum systems is preferable to qubit representations in some cases—see also [3]. Establishing qutrit states in quantum optics can be realized in different ways [2, 4, 5]. One of them is taking into account the three-optical-mode entanglement at a multiport beam splitter (tritter) [4]. Another possibility is based on exploiting a non-linear spontaneous parametric down-conversion (SPDC) process as a source of entangled linearly polarized photons [5]. In this case the polarization characteristics for the biphoton state can be represented by three components of a polarization vector.

Another key problem is measuring the qutrit state characteristics. Nowadays, a quantum state tomography approach to this is being discussed [6–8]. The basic idea of the method is to reconstruct the density matrix elements via a set of measurements (projections). The quantum tomography for low dimensional systems, e.g. for spin states, was first proposed in [9, 10]. It has been shown that the density matrix elements can be reconstructed from marginal distributions—for especially prepared diagonal elements of the density matrix—with the help of a unitary transformation. In quantum optics the density (or coherency) matrix elements can be expressed in terms of $SU(2)$ observables—Stokes parameters of the optical field [11, 12]. In fact, the spin state tomography approach is very close to the usual classical ellipsometric measurement technique [12, 13]. For the latter case the measured intensity of light $I(\chi, \delta) = \frac{1}{2}(S_0 + S_1 \cos(2\chi) + S_2 \sin(2\chi) \cos(\delta) + S_3 \sin(2\chi) \sin(\delta))$ depends on the $\delta$ and $\chi$ parameters introduced by the linear optical elements, i.e., phase plates and polarizers. Thus, we need at least four measurements of the Stokes parameters $S_j$ ($j = 0, 1, 2, 3$) for complete determination of the light polarization state. In the quantum domain the Stokes parameters $S_1, S_2, S_3$ do not commute with each other, and therefore the procedure for the measurement of these parameters should be clarified. In general, for a quantum optical field we are able to carry out measurements consecutively with any desired accuracy. But in this case a number of copies of the initial quantum state of the system are absolutely necessary for doing the measurements [6, 7].

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In an alternative operational approach, the measuring apparatus operates on the initial quantum state of the system and performs all measurements for non-commuting observables simultaneously with some accuracy determined by uncertainty relations. This approach becomes preferable in some cases, and results in unique information being obtained about quantum characteristics of the system, e.g. for the problem of optical phase measurements [14] and quantum polarization phase characteristic determination [15]. In particular, in [15] we consider a special multiport interferometer for simultaneous measurement of all polarization Stokes parameters of light. The quantum error of the measurement is determined by vacuum field fluctuations, and plays a principal role in this case, as it cannot be avoided due to the quantum nature of the optical field. Such an approach permits us to obtain complete information about the density matrix elements (i.e. coherency) and to evaluate the degree of polarization for light as well.

In this paper we develop a quantum theory for SU(3) polarization symmetry systems. The approach is based on three-mode interaction in quantum optics in general. We analyse the quantum phase and amplitude properties in a three-level system. In section 2 the mathematical description of the SU(3) polarization states for optical fields is presented. Section 3 is devoted to the problem of operational determination of non-diagonal elements of the coherency matrix and the degree of polarization, with the help of the Gell-Mann parameters of the optical field, by making simultaneous measurements by means of the twelve-port interferometer. The measurement of the amplitude characteristics of the three-mode optical field is considered in section 4.

2. Quantum description of SU(3) polarization for Bose systems

The quantum three-mode Bose system with SU(3) symmetry can be described in the Schwinger representation with Hermitian Gell-Mann operators \( \lambda_j, j = 0, 1, \ldots, 8 \) (cf [8]):

\[
\begin{align*}
\lambda_0 &= a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3, \\
\lambda_1 &= a_1^\dagger a_2 + a_2^\dagger a_1, \quad \lambda_2 = i(a_1^\dagger a_3 - a_2^\dagger a_3), \\
\lambda_3 &= a_1^\dagger a_3 - a_2^\dagger a_2, \\
\lambda_4 &= a_1^\dagger a_3 + a_2^\dagger a_1, \quad \lambda_5 = i(a_1^\dagger a_3 - a_2^\dagger a_3), \\
\lambda_6 &= a_1^\dagger a_3 + a_2^\dagger a_2, \quad \lambda_7 = i(a_1^\dagger a_3 - a_2^\dagger a_3), \\
\lambda_8 &= \frac{1}{\sqrt{3}}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3),
\end{align*}
\]

where \( a_j^\dagger \) (\( a_j \)), \( j = 1, 2, 3 \), are the photon annihilation (creation) operators. The operators \( \lambda_j \) defined above in equation (1) obey standard commutation relations for SU(3) algebra—see e.g. [8].

In expression (1a) the operator \( \lambda_0 \) determines the total number of photons; the operators \( \lambda_{1,2,3} \) represent the SU(2) subgroup of the SU(3) algebra. In quantum optics this subgroup corresponds to the polarization Stokes parameters \( S_{1,2,3} \) for the optical field, where the modes 1 and 2 are two linear (circular) polarization components of light [11]. The operators \( \lambda_{4,5} \) and \( \lambda_{6,7} \) in expressions (1c), (1d) characterize coupling between the first two modes (\( j = 1, 2 \)) and the third one (\( j = 3 \)), respectively. The operator \( \lambda_8 \) in equation (1e) is given as a combination of photon numbers for all modes of the quantum field.

Let us introduce the unit vector \( \vec{e} \) for a three-mode system in Hilbert space:

\[
\vec{e}a = \vec{e}_1 a_1 + \vec{e}_2 a_2 + \vec{e}_3 a_3
\]

(2)

where \( a \) is the annihilation operator for a three-mode field; \( \vec{e}_j \) (\( j = 1, 2, 3 \)) are orthogonal vectors obeying the condition

\[
\sum_{j=1}^3 |\vec{e}_j|^2 = 1.
\]

(3)

The relation (2) can be written as

\[
a = e_1^* e_1 + e_2^* e_2 + e_3^* e_3,
\]

(4)

where \( e_j^* \) are the projections of vector \( \vec{e} \).

The expressions (2)–(4) determine the decomposition of a three-mode optical field by analogy with the usual decomposition of elliptically polarized light with respect to two orthogonal (linear or circular) polarization components in quantum optics—cf [13]. However, only two orthogonal vectors fulfill the transversality condition for plane waves. Therefore the physical meaning of the \( e_j \) parameters in expressions (2)–(4) should be clarified for each specific polarization problem. For example, we also refer here to some problems of quantum optics when an additional (longitudinal) component of the polarization is present (see e.g. [16]) and our description can be useful.

Thus, we rewrite expression (4) for the three-mode problem in terms of the four parameters \( \theta, \phi, \psi_1, \psi_2 \) according to the SU(3) symmetry approach (see e.g. [8]):

\[
e_1 = e^{i\phi_1} \sin \theta \cos \phi, \quad e_2 = e^{i\phi_2} \sin \theta \sin \phi, \quad e_3 = \cos \theta,
\]

(5)

where we have the parameters \( \theta, \phi \in [0; \pi/2] \) and \( \psi_{1,2} \in [0; 2\pi] \).

The parameters \( \theta \) and \( \phi \) describe amplitude characteristics of the three-level quantum system. The phase properties of the quantum state of the optical field with SU(3) symmetry are determined by the parameters \( \psi_1 \) and \( \psi_2 \).

Let us consider the SU(3) polarization state for a coherent optical field:

\[
a|\alpha\rangle = \alpha|\alpha\rangle, \quad a_j|\alpha\rangle = \alpha_j|\alpha\rangle, \quad j = 1, 2, 3
\]

(6)

where \( |\alpha\rangle = |\alpha_1\rangle |\alpha_2\rangle |\alpha_3\rangle \) is the coherent state of the three-mode field. With the help of expressions (4)–(6) we obtain

\[
a = \sum_{j=1}^3 e_j^* a_j, \quad a_j = e_j a_j, \quad j = 1, 2, 3.
\]

(7)

In this paper we also examine a tripartite state that can be established as (cf [17])

\[
|\Psi\rangle_N = \frac{1}{\sqrt{N!}} (e_1 a_1^\dagger + e_2 a_2^\dagger + e_3 a_3^\dagger)^N |0\rangle.
\]

(8)
where \( |0\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_3 \) is a vacuum state; \( N = \langle \lambda_0 \rangle \) is the total number of particles. For the microscopic limit, when \( N = 1 \), we have a tripartite entangled qutrit W-state for a three-level system:

\[
|\Psi\rangle_3 = e_1|1\rangle_1 |0\rangle_2 |0\rangle_3 + e_2|0\rangle_1 |1\rangle_2 |0\rangle_3 + e_3|0\rangle_1 |0\rangle_2 |1\rangle_3. \tag{9}
\]

The state \( |\Psi\rangle_3 \) can be produced by using a tritter [4]. The maximally entangled state is realized for \( e_i = 1/\sqrt{3} \) (\( \theta = \arccos(1/\sqrt{3}) \)), but for some problems of quantum information the maximally entangled qutrits are not optimal—see e.g. [1]. In this paper we also consider non-symmetric qutrit W-states that can be obtained from equation (9) for \( \theta = \phi = \pi/4 \).

For \( e_i = 0 \) and \( e_j \neq 0 \) (i, j = 1, 2, 3, i \neq j) the qutrit state \( |\Psi\rangle_3 \) in equation (9) reduces to one of the three-qubit states |\Psi\rangle_{ijk}.

With the help of definitions (1) it is easy to obtain the following relations for the Gell-Mann parameter variances for the optical field in state \( |\Psi\rangle_N \):

\[
N\langle |\Delta \lambda_0 \rangle^2 |\Psi\rangle_N = 0,
\]

\[
N\langle |\Delta \lambda_j \rangle^2 |\Psi\rangle_N \leq \langle \alpha |(\Delta \lambda_j)^2 |\alpha \rangle, \quad j = 1, \ldots, 8 \tag{10}
\]

where the expressions \( \langle \alpha |(\Delta \lambda_j)^2 |\alpha \rangle \) represent the variances of the Gell-Mann parameters (1) for coherent state (6). The inequality in (10) characterizes non-classical properties of entangled states (8), (9) when the optical field fluctuations that correspond to the Gell-Mann parameter variances are suppressed below the level of the fluctuations for coherent states, i.e. the effect of squeezing occurs.

3. Degree of polarization; the \( SU(3) \) interferometer

Let us consider for the first time the problem of the degree of polarization for an optical field with \( SU(3) \) symmetry.

We start from the classical definition for a two-mode optical system. In particular, in the case of stochastic plane waves the degree of polarization \( P_2 \) can be represented as (see e.g. [13])

\[
P_2 = \left( 1 - \frac{4 \det(J_2)}{(Tr(J_2))^2} \right)^{1/2}, \tag{11}
\]

where \( J_2 \) is the 2 \times 2 coherency matrix. It is important that the \( P_2 \) parameter could be expressed in terms of the scalar invariants \( Tr(J_2) \), \( det(J_2) \) and \( Tr((J_2)^2) = (Tr(J_2))^2 - 2 det(J_2) \).

Alternatively, the degree of polarization \( P_2 \) defined in equation (11) can be rewritten in terms of Stokes parameters \( \langle S_j \rangle \) (j = 0, 1, 2, 3) for a two-mode optical field as

\[
P_2 = \left( \langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2 \right)^{1/2}/\langle S_0 \rangle^{1/2}. \tag{12}
\]

Although there are various definitions of the degree of polarization in quantum optics (we do not discuss them in this paper—see e.g. [18–20]), the expression (12) can be used in the quantum domain as well. In this case the \( \langle S_j \rangle \) variables in equation (12) represent expectation values of the Stokes operators \( S_j \).

Now we switch our attention to the case of a three-mode optical system (non-plane waves—cf [13]).

Quantum properties of the optical field with \( SU(3) \) symmetry can be described in this case using the following density (or coherency) matrix \( J_3 \):

\[
J_3 = \begin{pmatrix}
\langle a_1^\dagger a_1 \rangle & \langle a_1^\dagger a_2 \rangle & \langle a_1^\dagger a_3 \rangle \\
\langle a_2^\dagger a_1 \rangle & \langle a_2^\dagger a_2 \rangle & \langle a_2^\dagger a_3 \rangle \\
\langle a_3^\dagger a_1 \rangle & \langle a_3^\dagger a_2 \rangle & \langle a_3^\dagger a_3 \rangle
\end{pmatrix}. \tag{13}
\]

In general, the scalar invariants \( Tr(J_3^2) \) and \( Tr(J_3^2) \) arise in this case. In particular, they can be expressed as

\[
Tr(J_3^2) = P_3^2 (Tr(J_3))^3 + 3 det(J_3),
\]

\[
Tr(J_3^2) = \frac{(Tr(J_3))^2}{3} (1 + 2 P_3^2), \tag{14}
\]

where \( P_3 \) represents the degree of polarization for a three-mode optical field. With the help of the definition of Gell-Mann operators (1), the quantity \( P_3 \) can be represented as

\[
P_3 = \frac{\sqrt{3}}{2} \frac{\sum_{j=1}^{8} \langle \lambda_j \rangle^2}{\langle \lambda_0 \rangle}. \tag{15}
\]

Note that expression (15) for the degree of polarization \( P_3 \) for a three-mode optical system can also be considered as a generalization of the well known definition of the degree of polarization \( P_2 \) in equation (12).

The quantum system with \( SU(3) \) symmetry is completely polarized if and only if

\[
det(J_3) = 0, \quad P_3 = 1. \tag{16}
\]

It is easy to check that the states (6), (8) and (9) fulfil conditions (16).

Let us consider the procedure of measurement of the non-diagonal matrix elements of \( J_3 \) (see equation (13)). The schematic set-up for measurement of all phase dependent Gell-Mann parameters is shown in figure 1. At the input of the system we have three modes \( b_j \) (j = 1, 2, 3). The boxes \( B_j \) represent the balanced beam splitters or symmetric cloning machines (see e.g. [21]) used to produce the three modes \( a_j \) (j = 1, 2, 3) and their copies (clones) \( a_j' \) at the output. Then, each of the three-mode sets is transformed into two physically identical star-like interferometers denoted as \( I_1 \) and \( I_2 \) respectively. Thus, we obtain six Gell-Mann parameters at the output of the device in figure 1.

Let us precisely analyse the \( SU(3) \) twelve-port interferometer scheme (figure 2) for operational (simultaneous) measurement of the three Gell-Mann parameters \( \lambda_j \). The 100% efficiency detectors \( D \) are the devices used for measurement of the photon numbers \( N_{ij} = d_{ij}^\dagger d_{ij} \), where \( d_{ij} \) (\( d_{ij}^\dagger \)) are annihilation (creation) operators for the modes at the output of the interferometer. They can be represented as linear combinations of input fields \( a_j \) and vacuum modes \( V_j \) after the beam splitters (BS) in figure 2. The measurement procedure results in the detection of photon number differences \( N_{ij}^{i<} \), i, j = 1, 2, 3, i < j:

\[
N_{12}^{i<} = N_{12} - N_{21} = \frac{1}{2} \lambda_{12} + M_{12}, \tag{17a}
\]

\[
N_{13}^{i<} = N_{13} - N_{31} = \frac{1}{2} \lambda_{13} + M_{13}, \tag{17b}
\]

\[
N_{23}^{i<} = N_{32} - N_{23} = \frac{1}{2} \lambda_{23} + M_{23} \tag{17c}
\]

where the Gell-Mann parameters \( \lambda_{ij} \) are represented as (cf equations (1)).
The normally ordered operators $M_{ij}$ in equation (17) are proportional to the operators $a_j$ $(a_j^\dagger)$ and the vacuum modes $V_j$, $j = 1, 2, 3$, at the input of the interferometer. The average values of these operators fulfill the condition $\langle M_{ij} \rangle = 0$. From equations (18) it is easy to see that with the phase shifts $\phi_j = 0$ for interferometer $I_1$ we measure the Gell-Mann parameters $\lambda_1, \lambda_4$ and $\lambda_5$—equations (1), and if $\phi_1 = \frac{\pi}{4}, \phi_{2,3} = -\frac{\pi}{4}$ the measurement of $\lambda_2, \lambda_5$ and $\lambda_7$ is realized by interferometer $I_2$—see figure 1.

From equations (17), for the average values of the photon number difference $\langle N_{ij}^{(-)} \rangle$ and for the variances $\langle (\Delta N_{ij}^{(-)})^2 \rangle$, we obtain

$$\langle N_{ij}^{(-)} \rangle = \frac{1}{2}(\lambda_{jj}),$$

$$(\Delta N_{ij}^{(-)})^2 = \frac{1}{4}(\Delta \lambda_{ij})^2 + \frac{1}{2}(a_j^\dagger a_i + a_j a_i^\dagger),$$

where the condition $\langle M_{ij} \rangle = 0$ is taken into account.

The last two terms in equation (19b) are determined by the vacuum fluctuation contribution of the modes $V_j$ and characterize the lowest possible level of the variances considered when $\langle (\Delta \lambda_{ij})^2 \rangle = 0$.

Let us consider an ultimate case for figure 2. We assume the $\alpha_3$ mode to be the control field in the coherent state $|\alpha_3\rangle$ with the complex amplitude $\alpha_3 = |\alpha_3|e^{i\phi}$ ($\phi$ is the phase). The measured mean photon number differences $\langle N_{ij}^{(-)} \rangle$ are

$$\langle N_{12}^{(-)} \rangle = \frac{1}{2}(\lambda_1 \cos(\phi_2) - \lambda_2 \sin(\phi_2)),$$

$$\langle N_{13}^{(-)} \rangle = \frac{1}{2}(\alpha_3)(|q_1| \cos(\phi_1) - |p_1| \sin(\phi_1)),$$

$$\langle N_{23}^{(-)} \rangle = \frac{1}{2}|\alpha_3|^2(|p_2| \sin(\phi_3) + |q_2| \cos(\phi_3))$$

where $q_j = a_j^\dagger e^{-i\phi} + a_j e^{i\phi}$, $p_j = i(a_j^\dagger e^{-i\phi} - a_j e^{i\phi}), j = 1, 2$ are the Hermitian quadratures for two other modes.

Let us briefly discuss the properties of the degree of polarization $P_3$ in equation (15) in the case under consideration. In the limit of weak control field $|\alpha_3|^2 \ll 1$ we can measure quantum polarization properties for the two-mode optical field (for the bipartite qutrit state as well—cf [5]) by means of the scheme in figure 2. Note that the relation (15) for $P_3$ in this case will differ from the usual definition of the degree of polarization for a two-mode system—see equation (12) and [11]. In the other limit when $|\alpha_3|^2 \gg |a_j^\dagger a_j|$, the control field $a_3$ plays the role of a strong local oscillator field for simultaneous homodyne measurement of the quadratures of a two-mode optical system according to equations (20b) and (20c). In this case we have $P_3 \simeq 1$ from equation (15).

### 4. Measurements of the parameters $\theta$ and $\phi$

In this section we consider the problem of simultaneous measurement of diagonal elements of the matrix $J$ (equation (13)). That is, we focus our attention on the measurement of diagonal elements of the matrix $J$ in equation (17).

$$J^{\pm} = ( J \sigma_3 + \sigma_3 J )^{\pm}$$

In equation (17) $\sigma_3$ is the Pauli matrix and $J$ is the J-th moment of the light intensity $I_j = \langle a_j^\dagger a_j \rangle$ for simultaneous homodyne measurement of the quadratures of a two-mode optical system according to equations (20b) and (20c).

$$S_\theta = \sqrt{\hat{n}_3 \hat{n}_1 + \hat{n}_2}, \quad C_\theta = \sqrt{\hat{n}_1 \hat{n}_2 + \hat{n}_3}$$

where $\hat{n}_j = a_j^\dagger a_j$ is the photon number operator and we use equation (7) as well. To calculate the expectation values and variances for the amplitude operators we represent the operators $\hat{a}_j$ as $\hat{a}_j = \hat{a}_j + \hat{\Delta a}_j$ $(j = 1, 2, 3)$, where $\hat{a}_j$ is the mean (classical) value of the $j$-th photon number, $\hat{\Delta a}_j$ is the small fluctuation part of the corresponding operator with the properties $\langle \hat{\Delta a}_j \rangle = 0$, $\langle \hat{\Delta a}_j \hat{\Delta a}_j \rangle = 0$ ($j = 1, 2, 3, i \neq j$).
After some mathematical calculations of variances of detected relative amplitudes \(\langle \Delta S_{n}^2 \rangle, \langle \Delta C_{n}^2 \rangle\), we obtain
\[
\langle \Delta S_{n}^2 \rangle = \frac{n_1}{4(n_1 + n_2)^2} \left( \frac{n_2}{n_1} \sigma_1^2 + \frac{n_1}{n_2} \sigma_2^2 \right),
\]
\[
\langle \Delta C_{n}^2 \rangle = \frac{n_2}{4(n_1 + n_2)^2} \left( \frac{n_2}{n_1} \sigma_1^2 + \frac{n_1}{n_2} \sigma_2^2 \right),
\]
\[
\langle \Delta S_{n}^2 \rangle = \frac{1}{N^3} \left( \bar{n}_3 (\sigma_1^2 + \sigma_2^2) + (\bar{n}_1 + \bar{n}_2) \sigma_3^2 \right),
\]
\[
\langle \Delta C_{n}^2 \rangle = \frac{1}{N^3} \left( \bar{n}_3 (\sigma_1^2 + \sigma_2^2) + (\bar{n}_1 + \bar{n}_2) \sigma_3^2 \right)
\]
where \(N = \bar{n}_1 + \bar{n}_2 + \bar{n}_3\) is the total average number of photons. In relations (22) the variances in the photon numbers \(\sigma_j^2 \equiv \langle \Delta n_j^2 \rangle\) \((j = 1, 2, 3)\) characterize the terms additional to the mean values of the amplitude parameters in the quantum domain. In a quasi-classical limit \((N \gg 1)\), from equations (22) we have
\[
\langle \Delta S_{n}^2 \rangle \simeq \langle \Delta C_{n}^2 \rangle \simeq 0, \quad j = \phi, \theta, \quad (23)
\]
i.e. the role of fluctuations is not essential. For the three-mode Fock state we have \(\sigma_j^2 = 0\) and the conditions (23) are satisfied as well. The minimal (zero) value of the relative amplitude fluctuations can also be achieved for qubit states when one of the parameters \(e_j = 0\).

The equations (22) are violated for a small average number of photons \((\bar{n}_j \ll 1)\). In the case of the photon counting measurement the average value of some amplitude operator \(f(\bar{n})\) is represented as
\[
\langle f(\bar{n}) \rangle = \sum_{|n|} f(|n|) W(|n|)
\]
where \(W(|n|) \equiv W(n_1, n_2, n_3)\) is the joint probability of the set of eigenvalues \(n_1, n_2, n_3\) having the form
\[
W(|n|) = \left( \prod_{j=1}^{3} \frac{(a_j^* a_j)^{n_j} e^{-a_j^* a_j}}{n_j!} \right).
\]

For a weak field \((\bar{n}_j \ll 1)\) the term with \(n_1 = n_2 = n_3 = 0\) should be discarded in equation (24) and the sum in relation (24) should be renormalized with respect to the expression \(1 - \langle \exp(-\sum_j a_j^* a_j) \rangle\) —cf [14]. Finally, for the variances \(\langle \Delta S_j^2 \rangle, \langle \Delta C_j^2 \rangle\) we obtain
\[
\langle \Delta S_j^2 \rangle = \langle \Delta C_j^2 \rangle = \frac{1}{2} \sin^2 (2\phi),
\]
\[
\langle \Delta S_j^2 \rangle = \langle \Delta C_j^2 \rangle = \frac{1}{2} \sin^2 (2\theta).
\]

The minimal (i.e. zero) value of the fluctuations corresponds in equations (26) to the qubit state when some \(\bar{n}_j = 0\) (but not the denominator). The maximal value of the variances is \(\langle \Delta S_j^2 \rangle = \langle \Delta C_j^2 \rangle = \frac{1}{4}\) for non-symmetric qutrits \((\theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})\). The variances are \(\langle \Delta S_j^2 \rangle = \langle \Delta C_j^2 \rangle \approx 0.22\) for the maximally entangled qutrit state (9) when \(n_1 = n_2 = n_3\).

5. Conclusion

In this paper we have presented the quantum theory of SU(3) polarization for a three-mode optical field. In particular, we define the degree of polarization that can be connected with scalar invariants with respect to unitary transformations. We present a special twelve-port interferometer for operational reconstruction of non-diagonal elements of the coherency matrix \(A_j\) and for the measurement of the degree of polarization \(P_\theta\) as well. It is important that the scheme in figure 2 contains only linear optical elements—balanced beam splitters—as input ports for splitting the state for each input mode \(a_j\).

To describe amplitude properties of an optical field with SU(3) symmetry we introduce special variables (operators \(S_i\) and \(C_j\), \(j = \theta, \phi\)) that can be measured by a direct simultaneous photodetection procedure. We have shown that qutrit states are characterized by non-vanishing variances of these variables.

In this paper we do not present a procedure for measurement of the phases \(\psi_1\) and \(\psi_2\)—see equation (5). The operators for the phase parameters \(\psi_{1,2}\) can be introduced operationally by using the interferometer in figure 2 as well—cf [15].

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