Interaction of two polarization modes in a spatio-periodic nonlinear medium: generation of polarization-squeezed light and quantum non-demolition measurements of the Stokes parameters

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Abstract. The possibility of formation of polarization-squeezed light in the case of the interaction of two orthogonally polarized waves in a spatio-periodically Kerr-like medium is considered. In such a quantum polarization state of light the fluctuations of one of the Stokes parameters are less than in the coherent one. Both linear and nonlinear energy exchange between polarized modes are very important for the formation of the non-classically polarized light. For the first time we offer and analyse the procedure of the quantum non-demolition measurement (QND) of the Stokes parameters of light. Conditions of realization of the QND measurement are formulated in the general case.

Introduction

Highly efficient schemes for non-classical light generation based on distributed feedback (DFB) systems have been described in many papers recently (see e.g. [1]). The most promising realization of such a DFB could be established with nonlinear optical fibres of a special type, i.e. with fibres in which the dielectric constant varies periodically along the direction of propagation and with fibres with two (tunnelling-coupled) cores [2]. In fact, the formation of both quantum-squeezed states and sub-Poissonian statistics for propagating radiation and also the possibility of realization of quantum non-demolition measurements were proposed for these cases [1–3]. The physical cause of these phenomena is determined by the processes of interference and energy exchange (both linear and nonlinear) between two coupled waves.

There is some evidence to suggest that interaction of two orthogonally polarized modes in these types of optical fibres as well as in a spatially inhomogeneous (twisted) nonlinear optical fibres (see, for example, [4, 5]) results in generation of non-classically polarized light.

In fact, a new type of quantum state, so-called polarization-squeezed (PS) light, has recently been proposed to be used in precise polarimetry and ellipsometry (compare [6, 7] with [8]). Such light has smaller quantum fluctuations for one of the Stokes parameter than for the coherent state.

The possibility of generation of PS light is determined by the third-order nonlinearity of an anisotropic medium [7]. Another scheme which can also be associated with the...
reduction of fluctuations for the Stokes parameters is described in [9] where the suppression of fluctuations for the photon number difference (and/or sum) has been obtained.

An important problem which arises in connection with experimental observation of PS light is to select the procedure of measurement of the Stokes parameters by the quantum non-demolition (QND) technique. Such a QND experiment plays an essential role in quantum optics for detection of non-classical light (see, e.g., [3, 10–12] and especially more recent works [13], which introduced the best results so far on this problem). However, schemes proposed previously cannot be directly applied for solving the problem because they do not take into account the polarization characteristics of radiation.

Indeed, a very well known QND measurement gives us information about the photon number [12] or about the Hermitian field quadratures [3, 11]. These parameters can only be associated with one of the polarization components of the field under study. These are the Stokes parameters of the light that are to be measured for a complete analysis of the polarization state of the radiation. Here we are talking about the polarization state of light only in contrast with the case of complete determination of the state of the two modes of the light field in general. For this last case the set of Stokes parameters does not completely describe the quantum state of such a two-mode field.

All these questions are under study in this paper. The material of the paper is arranged as follows. In section 1 we consider the formation of the polarization-squeezed light in a nonlinear spatially inhomogeneous (twisted) optical fibre. The procedure of quantum non-demolition measurements of the Stokes parameters for light is presented in section 2. In the appendix we give some details of the mathematical calculations.

1. Non-classical polarization states

1.1. Basic equations

In this section we will show that the interaction of two orthogonally polarized modes in nonlinear spatially inhomogeneous optical media (i.e. DFB systems) leads to the generation of polarization-squeezed (PS) light with non-classical properties. Let us start our analysis with the classical equations.

We will describe the propagation of linearly polarized waves in a periodically inhomogeneous nonlinear optical fibre by truncated equations for the slowly varying complex amplitudes $A_x$ and $A_y$ (i.e. the components of polarization along the $x$ and $y$ axes, respectively) [5]:

$$\begin{align*}
-\imath \frac{dA_x}{dz} &= k_x A_x + 2\beta \cos k_0 z A_y + R \left\{ |A_x|^2 + \frac{2}{3} |A_y|^2 \right\} A_x + \frac{1}{3} R A^+_x A^+_y \\
-\imath \frac{dA_y}{dz} &= k_y A_y + 2\beta \cos k_0 z A_x + R \left\{ |A_y|^2 + \frac{2}{3} |A_x|^2 \right\} A_y + \frac{1}{3} R A^+_y A^+_x
\end{align*}$$

(1)

where $\beta$ is the linear coefficient of wave coupling, $R = 2k_0 n_2 / n_1$, $s$ is the nonlinear coefficient ($k_0 = \omega / c$), $n_1$ and $n_2$ are the linear and nonlinear refractive index, respectively; the $s$ parameter is determined by the field distribution over the fibre area; $k_{x,y}$ are the wavenumbers of the polarized modes along the $x$ and $y$ axes, respectively.

The transition to quantum equations for the nonlinear interaction of modes is made by replacing the complex amplitudes $A_i$ ($i = x, y$) in equations (1) by operators,

$$A_{x,y} \rightarrow \imath (2\pi \hbar / \varepsilon_0 V)^{1/2} a_{x,y} \quad A^+_{x,y} \rightarrow -\imath (2\pi \hbar / \varepsilon_0 V)^{1/2} a^+_{x,y}$$

(2)

† We are very grateful to the referee who drew our attention to this fact.
where \( V \) is the quantization volume, \( \hbar \) is the Planck constant and \( a_j (a_j^+) \) is a photon annihilation (creation) operator for the \( j \)th mode (\( j = x, y \)).

Let us substitute (2) into (1) writing the right-hand side of the latter equations in a normally ordered form and making the following operator replacements:

\[
\begin{align*}
 a_x(z) &= a_1(z) \exp \{0.5i(k_x + k_y + k_0)z\} \\
 a_y(z) &= a_2(z) \exp \{0.5i(k_x + k_y - k_0)z\}.
\end{align*}
\]  

As a result we obtain the equations for the \( a_{1,2}(z) \) operators

\[
\begin{align*}
 -i \frac{da_1}{dz} &= \frac{1}{2} \delta a_1 + \beta a_2 + i \tilde{R}(a_1^+ a_1 + \frac{2}{3} a_2^+ a_2) a_1 \\
 -i \frac{da_2}{dz} &= -\frac{1}{2} \delta a_2 + \beta a_1 + i \tilde{R}(a_2^+ a_2 + \frac{2}{3} a_1^+ a_1) a_2
\end{align*}
\]  

where \( \tilde{R} = 2\pi \hbar \omega R/\varepsilon_0 V \), but \( \delta = k_x - k_y - k_0 \) is the mismatch of the wavevectors. Below we will assume that \( \delta = 0 \). In equation (4) the terms proportional to \( e^{-2i\delta z} \) are omitted, i.e. we assume that \( k_0z \gg 1 \) (cf [4, 5]).

Equations (4) can also be derived from the Heisenberg operator evolution equation

\[
\dot{a}_j = \{a_j; H_{\text{int}}\} \quad (j = 1, 2)
\]  

with the interaction Hamiltonian

\[
H_{\text{int}} = \frac{\hbar c}{n_1} \{ \beta a_1^+ a_2 + \beta a_2^+ a_1 + \frac{1}{2} \tilde{R}(a_1^+ a_1^2 + a_2^+ a_2^2) + \frac{2}{3} \tilde{R} a_1^+ a_1 a_2^+ a_2 \}
\]  

where the derivative \( d/dt \) is replaced by \( -(c/n_1)d/dz \); \( c \) is the light velocity in vacuum.

The solution of (4) can be found in the form [1]

\[
a_{1,2}(z) = \frac{1}{\sqrt{2}} (C_1(z)e^{i\beta z} \pm C_2(z)e^{-i\beta z})
\]  

where \( C_{1,2}(z) \) are slowly varying operators. Putting expression (6) into equation (4) we find the nonlinear equations \( C_{1,2}(z) \):

\[
\frac{dC_{1,2}}{dz} = i \tilde{R} \left( \frac{2}{3} C_{1,2}^+ C_{1,2} + C_{2,1}^+ C_{2,1} \right) C_{1,2}.
\]  

The set of equations (7) is valid under the following approximations:

\[
1/k_0 \ll L_t < L_{nl}
\]  

where \( L_t = 1/\beta \) and \( L_{nl} = 1/\tilde{R}|a_1|^2 \) are the linear and nonlinear spatial scales of interaction of the polarized components.

The solution of equations (7) has the form

\[
C_{1,2}(z) = \exp \{i\sigma \left( \frac{2}{3} C_{1,2}^+ C_{1,2} + C_{2,1}^+ C_{2,1} \right) \} C_{1,2}.
\]  

Here and below \( C_{1,2} = C_{1,2}(z = 0) \) is the operator value at the input of the nonlinear medium; \( \sigma = Rz \). According to equations (9) and (6), the following commutation relations are valid for operators \( a_{x,y} \), \( a_{1,2} \) and \( C_{1,2}(z) \):

\[
[C_k(z); C_m^+(z)] = \delta_{km} \quad [a_i(z); a_j^+(z)] = \delta_{ij} \quad i, j = x, y, 1, 2 \quad k, m = 1, 2.
\]  

These relations yield the nonlinear equations for the polarization components.
1.2. The fluctuations of the Stokes parameters

Let us characterize the polarization state of the two-mode field under discussion by the Stokes operators (see, for example, [7, 8]):

\[
\begin{align*}
S_0(z) &= n_x(z) + n_y(z) \\
S_1(z) &= n_x(z) - n_y(z) \\
S_2(z) &= a_+^*(z)a_x(z) + a_-^*(z)a_y(z) \\
S_3(z) &= i(a_+^*(z)a_x(z) - a_-^*(z)a_y(z))
\end{align*}
\] (11a-d)

where \(n_j(z) = a_+^*(z)a_j(z)\) is the operator of photon numbers in the \(j\)th mode of the polarization. For any distance in the fibre the Stokes operators obey the commutation relations of the \(SU(2)\) algebra, namely

\[
\begin{align*}
[S_1(z); S_2(z)] &= 2iS_3(z) \\
[S_2(z); S_3(z)] &= 2iS_1(z) \\
[S_3(z); S_1(z)] &= 2iS_2(z).
\end{align*}
\] (12a-c)

The operator \(S_0(z)\) commutes with any operator \(S_j(z)\) \((j = 1, 2, 3)\).

The relations (12) reduce to the so-called Schrödinger–Robertson uncertainty relation (see [14]):

\[
\langle \Delta S_j^2(z) \rangle \langle \Delta S_k^2(z) \rangle \geq \langle (S_n(z))^2 \rangle / (1 - r_{jk}^2) \quad j, k, m = 1, 2, 3 \quad j \neq k \neq m \] (13)

where \(\langle \Delta S_j^2(z) \rangle = \langle S_j^2(z) \rangle - \langle S_j(z) \rangle^2\) is the variance of the fluctuations of the \(j\)th Stokes parameter; \(r_{jk}\) is the correlation coefficient between the Stokes parameters \(S_j\) and \(S_k\) which is defined by

\[
r_{jk} = ((S_j S_k) + \langle S_j S_j \rangle - 2 \langle S_j \rangle \langle S_k \rangle) / 2 \langle (\Delta S_j^2) (\Delta S_k^2) \rangle^{1/2}.
\] (14)

According to the form of (13) the complete set of Stokes parameters cannot be simultaneously and exactly measured in quantum optics.

It is well known that each polarization state of light may be represented by a single point in the space of the Cartesian coordinates \(S_1\), \(S_2\) and \(S_3\). The point with coordinates \(\langle S_1 \rangle, \langle S_2 \rangle\) and \(\langle S_3 \rangle\) on the sphere of radius \(\langle S_0 \rangle\) (i.e. the Poincaré sphere) corresponds to a completely polarized field. In such a three-dimensional space the coherent state of the field is displayed by the ball-shaped region of uncertainty of the Stokes parameters centred on the Poincaré sphere as shown in figure 1 [7].

![Figure 1. Transformation of a coherent state (1) into a polarization-squeezed state (2) on the Poincaré sphere. State (2) is the non-classical one of completely polarized light.](image)
We will assume that the interacting modes $a_{x,y}$ (and $a_{1,2}$, see equation (3)) at the input of the nonlinear medium are initially in a coherent state, i.e.

$$a_j|\alpha_j\rangle = |\alpha_j\rangle a_j$$  \hspace{1cm} (j = x, y, 1, 2)  \hspace{1cm} (15)

where $\alpha_j$ and $|\alpha_j\rangle$ are the eigenvalue and the eigenstate of the $a_j$ operator, respectively. The state of the whole field containing two modes is equal to $|\alpha\rangle = |\alpha_x\rangle |\alpha_y\rangle$. Taking the definitions (11) it is easy to obtain that

$$\langle \Delta S_j^2 \rangle = \langle n_j \rangle + \langle n_y \rangle$$  \hspace{1cm} (j = 1, 2, 3)  \hspace{1cm} (16)

$$\langle n_{x,y} \rangle = |\alpha_{x,y}\rangle^2 = |\alpha_{1,2}\rangle^2$$

where $|\alpha_{1,2}\rangle^2$ is the average photon number in coherent modes $a_{1,2}$ (equation (3)) at the input of the medium.

It follows from expressions (16) that the level of fluctuation of the Stokes parameters for two-mode coherent radiation is defined by the sum of the average photon numbers $\langle n_i \rangle$ and $\langle n_y \rangle$ in the modes under consideration.

Taking equation (13) into account, we can write conditions for the squeezed states of light in terms of the Stokes parameter variances:

$$\langle \Delta S_j^2(z) \rangle \geq |\langle S_0(z) \rangle|/(1-r_{jk}^2)^{1/2}$$

$$\langle \Delta S_k^2(z) \rangle \leq |\langle S_m(z) \rangle|/(1-r_{jk}^2)^{1/2}$$  \hspace{1cm} j, k, m = 1, 2, 3  \hspace{1cm} j \neq k \neq m  \hspace{1cm} (17)

where upper or lower signs are used.

From the physical point of view, the inequalities (17) mean that the spherical region of uncertainty (for coherent states (15) and (16)) of the Stokes parameter is transformed into an ellipsoid of uncertainty for squeezed states (17) (see also figure 1). This new type of non-classical polarization state has recently been called a polarization-squeezed (PS) state [7].

1.3. Polarization-squeezed light formation

Let us now turn to the analysis of statistical characteristics of the Stokes parameters (11) at the output of the nonlinear medium which we have considered in section 1.1. Suppose we have the coherent states for input operators $C_{1,2}(z = 0)$ at the input of the medium,

$$C_{1,2}|\xi_{1,2}\rangle = |\xi_{1,2}\rangle C_{1,2}$$  \hspace{1cm} (18)

where the input modes $a_{1,2}$ and $a_{x,y}$ are in coherent states (15) (see, for example, [1]), and we have that $\xi_{1,2} = (\alpha_1 \pm \alpha_2)/\sqrt{2}$ and $\alpha_{1,2}$ are the eigenvalues of the operators $C_{1,2}$ and $a_{1,2}$, respectively. We propose that $\xi_1 = \xi_2 = \alpha_1/\sqrt{2}$. Thus the input component of the polarization mode $\alpha_2$ (i.e. $\alpha_y$) is in the vacuum state $|0\rangle_y$, i.e. $\alpha_2 = \alpha_y = 0$ at $z = 0$.

Taking into account the relations (9) and also the fact that the initial modes are in a coherent state we obtain the average values for the Stokes parameters (11):

$$\langle S_0(z) \rangle = |\alpha_x|^2 + |\alpha_y|^2 = \langle n \rangle$$  \hspace{1cm} (19a)

$$\langle S_1(z) \rangle = \langle n \rangle \exp\left(\langle n \rangle (\cos(\sigma/6) - 1)\right) \cos 2\beta z$$  \hspace{1cm} (19b)

$$\langle S_2(z) \rangle = -\langle n \rangle \exp\left(\langle n \rangle (\cos(\sigma/6) - 1)\right) \sin \theta \sin 2\beta z$$  \hspace{1cm} (19c)

$$\langle S_3(z) \rangle = \langle n \rangle \exp\left(\langle n \rangle (\cos(\sigma/6) - 1)\right) \cos \theta \sin 2\beta z$$  \hspace{1cm} (19d)

where the phase $\theta = \Phi - k_0 z$ includes both the initial phase difference between the two orthogonally polarized waves (determined by $\Phi$) and the phase shift $k_0 z$ due to travelling
waves that deals with spatial differences for wavenumbers of two modes (see equation (3)), \(\langle n \rangle = |\alpha_i|^2\) is the input photon number.

The fact that the values of the average Stokes parameters (19b)–(19d) are proportional to exponential multipliers is purely due to the quantum effect. However, in a real situation we have \(|\sigma| \ll 1, \sigma^2(n) \ll 1\) and so \(\exp(n)\cos(\sigma/6) - 1\) \(\approx 1\).

In equations (19b)–(19d) the oscillating terms are coupled with both an energy exchange between the interacting modes (due to \(\beta \neq 0\)) and the phase difference \(\theta\). It is obvious that the Stokes parameters \(\langle S_2 \rangle = \langle S_3 \rangle = 0\), but \(\langle S_1 \rangle\) has a maximum, when \(2\beta z = 2m\pi\) \((m = 1, 2, 3, \ldots)\).

It should be noted that the polarization degree \(P = (\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2)^{1/2}/\langle S_0 \rangle\) does not depend on the coupling coefficient \(\beta\) and the phase \(\theta\). It is equal to

\[
P = \exp(-n\sigma^2/72).
\]

In a real situation we have \(P = 1\) with the accuracy which has been used to obtain the results of the previous section (see below).

The calculations lead to the following expressions for variances \(\langle \Delta S_{2,3}^2(z) \rangle\):

\[
\langle \Delta S_2^2(z) \rangle = \langle n \rangle \{1 + \chi^2 \sin^2 \theta + \chi \sin 2\theta\} \\
\langle \Delta S_3^2(z) \rangle = \langle n \rangle \{1 + \chi^2 \cos^2 \theta - \chi \sin 2\theta\}
\]

where we introduce an effective nonlinear parameter \(\chi = \psi_{\text{eff}} = \psi \cos 2\beta z\) (\(\psi = \sigma(n)/6 = \tilde{R}(n)z/6\)). In the general case the variances \(\langle \Delta S_{2,3}^2(z) \rangle\) from (21a) and (21b) depend on the \(\chi\) parameter, i.e. on the parameters \(\psi\) and \(2\beta z\), and also on the phase \(\theta\). Equation (21b) can also be obtained from equation (21a) by replacing the phase \(\theta \to \theta - \pi/2\). It arises from the operators \(S_2\) and \(S_3\) being quadrature components with respect to the operator \(\alpha_j^\dagger(z)\alpha_j(z)\).

The 3D dependences of the variances (21a) and (21b) are shown in figure 2. In general, it is obvious from figures 2(a) and (b) that the normalized variances of the Stokes parameters \(\sigma_{2,3}^2 = \langle \Delta S_{2,3}^2(z) \rangle/\langle n \rangle\) demonstrate the oscillation behaviour with nonlinear parameter \(\chi\) and phase \(\theta\). In particular, when \(\chi = 2, \theta \approx 1.2\) rad the minimal value of \(\sigma_2^2\) corresponds to the maximal value of \(\sigma_3^2\). Thus, if the value \(\sigma_2^2\) or \(\sigma_3^2\) is smaller than the coherent noise level \(\sigma_j^2 = 1\), then polarization-squeezed light is formed.

The extreme values of the expressions (21) are obtained from the magnitude of the phase \(\theta\) that satisfies the relation \(\tan 2\theta = -2/\chi\). For this case we have the following

![Figure 2](image-url)

**Figure 2.** 3D dependences for normalized variance \(\sigma_j^2 = \langle \Delta S_j^2(z) \rangle/\langle n \rangle\) \((j = 2, 3)\) of the Stokes parameters \(S_j(z)\) (a) and \(S_3(z)\) (b) from the effective nonlinear parameter \(\chi\) and phase \(\theta\). The value \(\sigma_j^2 = 1\) corresponds to the coherent level of the Stokes parameter variances.
minimal and maximal values of the variances of the Stokes parameters:

\[
\begin{align*}
\langle \Delta S_2^2(z) \rangle_{\text{min}} &= \langle n \rangle \left(1 + \frac{\chi^2}{4}\right)^{1/2} - |\chi|/2 \quad (22a) \\
\langle \Delta S_2^2(z) \rangle_{\text{max}} &= \langle n \rangle \left(1 + \frac{\chi^2}{4}\right)^{1/2} + |\chi|/2. \quad (22b)
\end{align*}
\]

From figure 3 it is obvious that the minimal variance \(\langle \Delta S_2^2(z) \rangle_{\text{min}}\) (equation (22a)) decreases with the nonlinear parameter \(|\chi|\). The equal values of the variance correspond to the same magnitudes of \(|\chi| = |\psi| \cos 2\beta z\). For the latter we have large values if \(\cos 2\beta z = \pm 1\).

The Stokes parameter variances are the same as for coherent states (see equation (5)) at the output of the medium when first \(2\beta z = \pi/2 + \pi m \ (m = 0, 1, 2, \ldots)\) and second, the effective nonlinear parameter \(|\chi| = 0\).

![Figure 3](image)

**Figure 3.** The minimum normalized variance \(\sigma_{j,mn}^2 = \langle \Delta S_j^2(z) \rangle_{\text{min}}/\langle n \rangle\) of the Stokes parameter \(S_j(z) \ (j = 2, 3)\) as a function of both the nonlinear parameter \(\psi\) and linear coupling coefficient \(2\beta z\).

The product of the variances (22a) and (22b) is minimal for the case considered above, i.e.

\[
\langle \Delta S_2^2(z) \rangle_{\text{min}} \langle \Delta S_3^2(z) \rangle_{\text{max}} = \langle n \rangle^2 \quad (23)
\]

and such an ultimate product corresponds to an ideal PS state. In this case the correlation coefficient (14) for the Stokes parameters \(S_2\) and \(S_3\) is equal to zero, i.e. \(r_{23} = 0\).

The minimal value of the expression (21a) reduces to the magnitude

\[
\langle \Delta S_2^2(z) \rangle_{\text{min}} = \langle n \rangle \sin^2 \theta \quad (24a)
\]

when \(\chi = -\cot \theta\). In this case the variance of the Stokes parameter \(S_3\) is maximal:

\[
\langle \Delta S_3^2(z) \rangle_{\text{max}} = \langle n \rangle (1 + \cos^2 \theta + \cot^2 \theta) \quad (24b)
\]

Within the limits when \(\langle \Delta S_2^2(z) \rangle_{\text{min}} \to 0\), i.e. \(\langle \Delta S_3^2(z) \rangle_{\text{max}} \to \infty\) the uncertainty relation (13) is obeyed. However, the small value of the phase \(\theta\) can be obtained only when nonlinear parameter \(\chi\) is very large (\(\chi = -\cot \theta\)). In this case the expressions (21a) and (21b) can be written in the form

\[
\begin{align*}
\langle \Delta S_2^2(z) \rangle_{\text{min}} &= \langle n \rangle / (1 + \chi^2) \quad (25a) \\
\langle \Delta S_3^2(z) \rangle_{\text{max}} &= \langle n \rangle (1 + \chi^2 + \chi^2/(1 + \chi^2)) \quad (25b)
\end{align*}
\]

It is easy to see that we have monotonic suppression of the variance \(\langle \Delta S_2^2 \rangle\) over the nonlinear parameter \(\chi\) (see also figure 3).
Let us consider the fluctuations of the $S_1$ Stokes parameter. In the same approximation, in which expressions (21) have been obtained, we find the following expression for the variance $\langle \Delta S_1^2(z) \rangle$:

$$\langle \Delta S_1^2(z) \rangle = \langle n \rangle \{ 1 + \psi^2 \sin^2 2\beta z \}. \quad (26)$$

It is interesting to note that the value of $\langle \Delta S_1^2(z) \rangle$ always remains larger than the same one for the coherent state, excluding the distance $z$ which satisfies the condition $2\beta z = \pi m$ ($m = 0, 1, 2, \ldots$).

2. Quantum non-demolition measurements of the characteristics of polarized light

2.1. General conception

Now we switch our attention to the problem of quantum non-demolition (QND) measurements of the Stokes parameters. The conditions for realization of such measurements can be formulated in a general form. We define $S_m$ as a non-demolished measured (called a signal) Stokes parameter, $S_p$ as a measuring (probe) parameter, and finally $S_i$ as an auxiliary parameter, where $m, p, i = 1, 2, 3, m \neq p \neq i$. First we explain the role of the Stokes parameters in the measurement process.

![Figure 4: The general scheme of the $S_m$ Stokes parameter QND measurement; $S_{m,p,i}^{\text{in}}$ and $S_{m,p,i}^{\text{out}}$ ($m, p, i = 1, 2, 3, m \neq p \neq i$) are the values of the Stokes parameters at the input and at the output of the QND apparatus, respectively.](image)

In general, for the QND measurement it is assumed (see figure 4) that a measured Stokes parameter $S_m$ interacts with the probe parameter $S_p$ in some physical system (we will call it a QND apparatus) [10]. The following basic conditions defining the non-demolition measurement procedure have to be taken into account.

First, the non-demolition measurement of the Stokes parameter $S_m$ is characterized by a correlation coefficient $K_1$ (see [3, 11]):

$$K_1 = \frac{|\langle S_m^{\text{in}} S_m^{\text{out}} \rangle + \langle S_m^{\text{out}} S_m^{\text{in}} \rangle - 2 \langle S_m^{\text{in}} \rangle \langle S_m^{\text{out}} \rangle|^2}{4 \langle (\Delta S_m^{\text{in}})^2 \rangle \langle (\Delta S_m^{\text{out}})^2 \rangle}. \quad (27)$$

The numerator in (27) takes into account the quantum coupling between the input value of $S_m^{\text{in}}$ and the output value of $S_m^{\text{out}}$ and describes the noise reduction for the measured Stokes parameter. In the ideal case of the QND measurement it is evident that the Stokes parameter $S_m^{\text{in}}$ is conserved, i.e. $S_m^{\text{in}} = S_m^{\text{out}}$, and therefore $K_1 = 1$.

Second, the reliability with which the measured value of $S_m^{\text{in}}$ can be observed under detection of $S_p^{\text{out}}$ is related to the second measurement correlation coefficient $K_2$ which is defined as

$$K_2 = \frac{|\langle S_m^{\text{in}} S_p^{\text{out}} \rangle + \langle S_p^{\text{out}} S_m^{\text{in}} \rangle - 2 \langle S_m^{\text{in}} \rangle \langle S_p^{\text{out}} \rangle|^2}{4 \langle (\Delta S_m^{\text{in}})^2 \rangle \langle (\Delta S_p^{\text{out}})^2 \rangle}. \quad (28)$$

In the ideal case of the $S_m^{\text{in}}$ measurement there is linear coupling between the $S_p^{\text{out}}$ output probe and the $S_m^{\text{in}}$ input Stokes parameters, i.e. $S_p^{\text{out}} = \lambda S_m^{\text{in}}$. Therefore we have that $K_2 = 1$, ...
where $\lambda_2$ is the QND gain. However, in a real situation it is difficult to satisfy the criteria (27) and (28) simultaneously. Therefore, such a type of QND measurement turns out to be always non-ideal (see below).

Third, the additional conditions for the QND measurements under consideration arise due to the demand of the preparation of the quantum state described by the Stokes parameters. Thus, additional correlation coefficients have to be taken into account to describe the polarization characteristics of the field at the output of the QND apparatus. Let us introduce such correlation coefficients between the Stokes parameter $S^\text{out}_i$ and the two other Stokes operators $S^\text{out}_m$ (measured) and $S^\text{out}_p$ (probe) as follows (see also (14)):

$$R_1 = \frac{|\langle S^\text{out}_p S^\text{out}_i \rangle + \langle S^\text{out}_i S^\text{out}_m \rangle - 2\langle S^\text{out}_i \rangle \langle S^\text{out}_m \rangle|^2}{4\langle (\Delta S^\text{out}_m)^2 \rangle \langle (\Delta S^\text{out}_i)^2 \rangle}$$

(29a)

$$R_2 = \frac{|\langle S^\text{out}_p S^\text{out}_i \rangle + \langle S^\text{out}_i S^\text{out}_p \rangle - 2\langle S^\text{out}_i \rangle \langle S^\text{out}_p \rangle|^2}{4\langle (\Delta S^\text{out}_p)^2 \rangle \langle (\Delta S^\text{out}_i)^2 \rangle}$$

(29b)

In the ideal QND measurement case the quantum noise of the $S_i$ Stokes parameter does not destroy the measuring procedure and the correlation coefficients (29a) and (29b) of the output Stokes parameters are equal to zero ($R_{1,2} = 0$). From the physical point of view, this condition means that the measurement process must be separated from destructive backaction of the $S_i$ observable [10]. Furthermore, one more correlation coefficient $R_3$ (coupling the probe ($S^\text{out}_p$) parameter and measured ($S^\text{out}_m$) parameter) could be considered for the case. So, we introduce the coefficient $R_3$ in the form

$$R_3 = \frac{|\langle S^\text{out}_p S^\text{out}_m \rangle + \langle S^\text{out}_m S^\text{out}_p \rangle - 2\langle S^\text{out}_m \rangle \langle S^\text{out}_p \rangle|^2}{4\langle (\Delta S^\text{out}_p)^2 \rangle \langle (\Delta S^\text{out}_m)^2 \rangle}.$$  

(29c)

This presentation characterizes a special condition for the preparation of a quantum system at the output of the QND apparatus in terms of the Stokes parameters. For a ‘good’ QND measurement procedure we need a complete correlation of the above-introduced parameters and so we also have $R_3 = 1$.

Thus, the evaluation of the correlation coefficients (27)–(29) is absolutely necessary to highlight the possibility of realizing the QND measurement procedure for any Stokes parameter. We will specify them for several schemes of measurement in the following sections.

### 2.2. QND measurement of the Stokes parameter $S_1$

Here we assume that the Stokes parameter $S^\text{out}_m = S^\text{in}_1$ can be measured either by the probe parameter $S^\text{out}_p = S^\text{out}_3$ (in this case $S_i = S_2$) or by $S^\text{out}_p = S^\text{out}_2$ (in this case $S_i = S_3$) without demolition.

In quantum optics the procedure of QND measurement is usually based on the process of mixing of two modes ($a_1$ and $a_2$) in the QND apparatus. In the measuring process one mode is used as a probe beam (we denote it by index 2) to estimate the characteristics of the second mode (the signal mode with index 1) [3]. The linear system introduced in the optical scheme of the QND measurement is very important for the performance of the QND measurement requirements considered above. For our case the linear system has to enable us to mix up the two orthogonally polarized modes. It is obvious that nonlinear interaction of the modes is also important [11].

To realize the necessary linear coupling between the measured Stokes parameter $S^\text{in}_i$ and the probe one, for example $S^\text{out}_3$, we consider using an optical phase plate (with
phase retardation $\Phi$) and also a linear spatially periodical optical fibre. We will describe the propagation of the two orthogonally polarized modes in such a system by the linear transformation for the operators $a_\pm$ and $a_y$ in the form

$$a_{i}^{\text{out}} = e^{-i\Phi} \cos \beta z a_{i}^{\text{in}} + i \sin \beta z a_{y}^{\text{in}},$$

$$a_{y}^{\text{out}} = \cos \beta z a_{y}^{\text{in}} + e^{i\Phi} \sin \beta z a_{i}^{\text{in}}$$

where $\Phi$ the phase of the mode $a_\pm$ after propagation through the phase plate and $a_{i,3}^{\text{in}}$ ($a_{i,3}^{\text{out}}$) are the values of the operators at the input (output) of the linear system. It should be noted that the operators (30) obey the commutation relations (10).

By substitution of (30) in expressions (11) we have for the Stokes parameters at the output of the QND apparatus,

$$S_{0}^{\text{out}} = S_{0}^{\text{in}}$$

$$S_{2}^{\text{out}} = (\cos \Phi S_{2}^{\text{in}} - \sin \Phi S_{3}^{\text{in}})$$

$$S_{1}^{\text{out}} = S_{1}^{\text{in}} \cos g - (\cos \Phi S_{3}^{\text{in}} + \sin \Phi S_{2}^{\text{in}}) \sin g$$

$$S_{3}^{\text{out}} = \cos g (\cos \Phi S_{3}^{\text{in}} + \sin \Phi S_{2}^{\text{in}}) + \sin g S_{1}^{\text{in}}.$$ 

The value $g = 2\beta z$ defines the efficiency of the linear transformation of the Stokes parameters and $S_{j}^{\text{in}}$ ($S_{j}^{\text{out}}$) ($j = 1, 2, 3$) is the Stokes parameter at the input (output) of the QND apparatus. It is easy to see that for the case of the phase value $\Phi = 2\pi n$ ($n = 0, 1, 2, \ldots$) there is a linear coupling between the measured $S_{j}^{\text{in}}$ and the probe $S_{j}^{\text{out}}$ Stokes parameters. This fact follows from equations (31),

$$S_{0,2}^{\text{out}} = S_{0,2}^{\text{in}}$$

$$S_{1}^{\text{out}} = \lambda_{1} S_{1}^{\text{in}} - \lambda_{2} S_{3}^{\text{in}}$$

$$S_{3}^{\text{out}} = \lambda_{1} S_{3}^{\text{in}} + \lambda_{2} S_{1}^{\text{in}}$$

where $\lambda_{1} = \cos g$, $\lambda_{2} = \sin g$.

To describe the QND measurement of the $S_{1}$-parameter let us first consider the behaviour of the coefficients (27) and (28) (for the case $m = 1, p = 3$) which characterize the performance of the linear system as a QND apparatus.

From expressions (32b) and (32c) the relevant correlation coefficients (27), (28) are obtained in the form

$$K_{1,2} = \left(\lambda_{1,2} V_{1} + r_{13} \lambda_{2,1} V_{3} \mp 2r_{13} \lambda_{1,2} \sqrt{V_{1} V_{3}} \right) / (\Delta S_{1}^{\text{out}})^{2}$$

$$\langle (\Delta S_{1,3}^{\text{out}})^{2} \rangle = \left(\lambda_{1,2} V_{1} + \lambda_{2,1} V_{3} \mp 2r_{13} \lambda_{1,2} \sqrt{V_{1} V_{3}} \right)$$

where $r_{13}$ is the correlation coefficient between two parameters $S_{1}^{\text{in}}$ and $S_{3}^{\text{in}}$ given by (14) and we introduced $V_{1,3} = \langle (\Delta S_{1,3}^{\text{in}})^{2} \rangle$. According to equation (33) the correlation between the Stokes parameters $S_{1}^{\text{in}}$ and $S_{3}^{\text{in}}$ gives rise to added noise in the $S_{1}^{\text{in}}$ measuring process. It relates to the non-commutativity of the Stokes operators (see expressions (11) and (12)). Only for a specially prepared quantum polarization state can one realize the case of $r_{13} = 0$. Various aspects of the problem will be discussed below and also in the appendix.

Let us now consider the two limiting expressions for the correlation coefficients (33).
(i) Weak linear energy exchange limit. For the case $\lambda_2 \rightarrow 0$ (i.e. $|\lambda_2| = \sin g \approx g \ll 1$) one can obtain from expression (33),

$$K_1 \simeq 1$$  \hspace{1cm} (34a)

$$K_2 \approx \left(1 + \frac{\lambda_2^2 V_3}{\lambda_1^2 V_1} + 2 \frac{\lambda_1}{\lambda_2} r_{13} (V_3/V_1)^{1/2}\right)^{-1} + r_{13}^2.$$ \hspace{1cm} (34b)

As follows from relations (34) we have an approximately non-demolition value of the $S^{in}_1$ parameter due to condition $K_1 \simeq 1$. However, at the same time we must require the following inequalities to be fulfilled for it to evade measurement:

$$V_3 \ll (\lambda_2/\lambda_1)^2 V_1 \approx g^2 V_1$$ \hspace{1cm} (35a)

$$r_{13}^2 \ll 1.$$ \hspace{1cm} (35b)

Since $g \ll 1$ the condition (35a) means that the probe Stokes parameter variance $V_3$ at the linear input system must be considerably less than $V_1$ of the measured quantity. We should also require the absence of correlation between $S^{in}_1$ and $S^{in}_3$ for a ‘good’ QND measurement realization. In other words, in this case the quantum noise has not been transformed from the probe Stokes parameter to the measured Stokes parameter and it has to be isolated from backaction of the probe quantity noise.

In figure 5 the qualitative picture for the ‘areas of uncertainty’ of the QND measurement of the $S^{in}_1$ Stokes parameter is displayed. The projection of the ‘uncertainty volume’ on the $(S^1, S^3)$-plane (cf figures 5 and 1) is depicted. If condition (35a) is obeyed, we have ‘the ellipsis of uncertainty’ with suppressed values of the variance $V_3$ at the input of the linear system. The limiting case considered by us above (see expressions (34a) and (34b)) is shown in figure 5(b). It is evident that there is a slight inaccuracy of the $S^{in}_1$ Stokes parameter measurement characterized by $\Delta S^{err}_3$ quantity when the detection of the $S^{out}_3$ parameter occurs (see also (34b)).

![Figure 5](image)

**Figure 5.** Qualitative picture of change of the $S_1$ Stokes parameter variance in the process of the QND measurement. The variances are shown: (a) at the input of the linear medium (ellipses of squeezing); (b) for the case of a small inaccuracy $\Delta S^{err}_3$ of the $S^{in}_1$ measurement; (c) for the case of a small demolition $\Delta S^{meas}_1$ in the process of the $S^{in}_1$ measurement.

Now let us analyse the behaviour of the $R_{1,2,3}$ correlation coefficients at the output of the QND apparatus. The ultimate conditions arising owing to this behaviour will also be displayed. Substituting (32) for (29a) and (29b) we get

$$R_{1,2} = \frac{r_{12}^2 + (\lambda_2/\lambda_1)^2 V_3/V_1 r_{23}^2 + 2 r_{12} r_{23} (\lambda_2/\lambda_1)^2 (V_3/V_1)^{1/2}}{1 + (\lambda_2/\lambda_1)^2 V_3/V_1 + 2 (\lambda_2/\lambda_1)^2 (V_3/V_1)^{1/2} r_{13}}$$ \hspace{1cm} (36)
where \( r_{12} \) and \( r_{23} \) are the correlation coefficients between the corresponding Stokes parameters at the input of the QND apparatus (see equation (14)).

When the conditions (35a) and (35b) are satisfied expressions (36) are simplified and we have

\[
R_1 \approx R_2 \approx r_{12}^2. \tag{37}
\]

Relation (37) means that in the limit of (35a) and (35b) considered above, the correlations between the \( S_{2}^{\text{out}} \) and \( S_{1,3}^{\text{in}} \) Stokes parameters at the output of the QND apparatus depend entirely on their values at the input of the discussed device.

Thus for a ‘high-quality’ QND measurement of the \( S_1^{\text{in}} \) Stokes parameter, small correlations between the input Stokes parameters are required, i.e.

\[
r_{12}^2 \ll 1. \tag{38}
\]

As will be shown in the appendix, this requirement is satisfied by using an anisotropic nonlinear medium placed just before the linear system.

Finally, let us briefly consider the role of the correlation coefficient \( R_3 \) (see equation (29c)). It is easy to show that fulfilment of the conditions (35) results in the demand of an ideal measurement \( R_3 \approx 1 \) at the same time. Thus the coefficient \( R_3 \) introduces no new limitation in the measurement procedure that is considered by us and so, the coefficient \( R_3 \) can practically be ignored.

The principal set-up for the discussed QND measurement of the \( S_1^{\text{in}} \) parameter is shown in figure 6. First, the initial radiation propagates through a medium with an anisotropic cubic nonlinearity, \( NL; S_{1b} \) and \( S_{3b} \) are the values of the Stokes parameters at the input of the QND apparatus. Then they are going into the linear system \( L \).

The detailed picture of propagation of the two orthogonally polarized modes \( b_x \) and \( b_y \) in the nonlinear optical medium under consideration is analysed in the appendix. There it is shown that the polarization-squeezed (PS) light, formed in the nonlinear medium (figure 6), satisfies the QND measurement conditions (35a) and (35b) for the Stokes parameter \( S_1^{\text{in}} \). At the same time the \( S_1 \) value, i.e. the difference of the photon numbers for two modes, is conserved during propagation of the field through the anisotropic nonlinear medium:

\[
S_{1b} = S_1^{\text{in}}. \tag{39}
\]

The condition (35a) for realization of the QND measurement of the \( S_1^{\text{in}} \) parameter leads to the demand that the polarization-squeezed light has to be at the input of the linear system. This fact takes into account that the variance of the measured Stokes parameter \( V_1 = \langle \Delta S_1^2 \rangle = \langle (\Delta S_1^{\text{in}})^2 \rangle \) corresponds to the coherence level of fluctuations at the entrance to the QND apparatus:

\[
V_3 \ll g^2 (|\beta_x|^2 + |\beta_y|^2) \tag{40}
\]

where \( |\beta_x|^2 \) and \( |\beta_y|^2 \) are the input average photon numbers, i.e. \( |\beta_x|^2 = |\beta_y|^2 = \langle n \rangle; g \ll 1. \)
Using expression (A5) from the appendix for the extremal value of the variance \(\langle (\Delta S_{in}^3)^2 \rangle\) of the probe Stokes parameter, we can rewrite the inequality (40) as a condition for nonlinear phase shift:

\[
\Psi_{nl} \gg 2 \cot 2g \approx 1/g .
\]  

Expression (41) has a simple physical meaning. The efficiency of the nonlinear interaction of the orthogonally polarized modes, which is defined by the nonlinear phase shift \(\Psi_{nl}\), must be greater than the corresponding efficiency of the energy exchange between two modes described by the parameter \(g\) of the linear system \(L\).

For ideal squeezing (see equation (A5)) the decreasing of variance \(V_3\) is accompanied by the increasing of variance \(V_2\) in the conjugate Stokes parameter \(S_2\). Thus, the precise measurement of the Stokes parameter \(S_{in}^1\) is realized in the anisotropic cubic-nonlinearity medium due to a previous redistribution of quantum fluctuations from the probe parameter \(S_{in}^3\) to the conjugate parameter \(S_{in}^2\). The latter is isolated from the measuring process (see expression (32)) in the linear system (figure 6).

(ii) High linear energy exchange limit. Now let us consider the other limiting case, when \(\lambda_1 \to 0\) (i.e. \(g = 2\beta z \to \pi/2\)). As a result we obtain from the relation (33):

\[
K_1 \approx \left(1 + \frac{\lambda_2^2 V_3}{\lambda_1^2 V_1} - \frac{2 \lambda_2}{\lambda_1} \left(V_3/V_1\right)^{1/2}\right)^{-1} + r_{13}^2 . \quad (42a)
\]

\[
K_2 \approx 1 . \quad (42b)
\]

In this case the QND apparatus is ‘tuned’ to measure the quantity \(S_{in}^3\) (see equation (42b)). However, there is a small demolition of the Stokes parameter \(S_{in}^1\) (this fact comes from (42a)). The situation is illustrated in figure 5(c), where the \(\Delta S_{in}^{\text{meas}}\) quantity just characterizes a small demolition of the \(S_{in}^1\) Stokes parameter caused by the measurement procedure. The demolition of the measured Stokes parameter could be negligible in the case when the correlation coefficient \(K_1 \to 1\) and in accordance with (42a) the following requirements are valid:

\[
V_3 \ll (\lambda_1/\lambda_2)^2 V_1 \quad (43a)
\]

\[
r_{13}^2 \ll 1 . \quad (43b)
\]

Although there is a difference between the two conditions (35a) and (43a), this is not important. For example, the requirements (35a) and (43a) actually coincide if we assume \(\lambda_1^2 = \cos^2 g = \cos^2(\pi/2 - g_0) = \sin^2 g_0 \approx g_0^2\), where the linear parameter \(g_0 \ll 1\). Thus, all the results of the previous analysis should be applied to the present case as well.

The consideration of the intermediate cases for values of the linear parameter \(g = 2\beta z\) characterizing the system \(L\) (figure 6), is not interesting for our analysis because they do not lead to new physical results (cf [3]). But it is important to emphasize that the QND measurement of the Stokes parameter \(S_{in}^3\) is not ideal because the coefficients (34) and (42) are not equal to unity simultaneously. The physical explanation of such non-ideal quantum measurements can be connected with the non-commutativity of the operator of the measured parameter \(S_1\) with one of the other Stokes parameters \(S_{2,3}\) from (12) (see also [10]).

In our forthcoming papers we will show that for an ideal QND measurement it is important to use a special scheme of four-wave mixing for two polarized modes. In this case the measurement of one set of Stokes parameters (that corresponds to one pair of polarization modes) is realized via the detection of the Stokes parameters of another pair of polarized modes.
Correlation coefficients $(a) K_1$ and $(b) K_2$ as a function of two parameters, the nonlinear phase shift $\Psi_{nl}$ and the linear coupling parameter $g$. The magnitudes of parameters used for calculations are: the extreme phase value $\phi = -0.5 \arctan(2/\Psi_{nl})$ and coefficients of the nonlinear medium’s anisotropy $\gamma = (\gamma_1 + \gamma_2)/2$.

Results. The 3D dependences (see equation (33)) of the correlation coefficients $K_1$ and $K_2$ as a function of both the nonlinear phase shift $\Psi_{nl}$ in the anisotropic medium $NL$ and the linear coupling coefficient $g$ in the linear system $L$ are shown in figures 7(a) and (b). We also assume that the condition (35b) is valid, i.e. $r_{13} = 0$ and the variance of the Stokes parameter $S_{in}$ at the entrance to the linear system is determined by the expression (A5).

Comparing figures 7(a) and (b) one can easily see that the correlation coefficients $K_1$ and $K_2$ do not take values equal to unity at the fixed magnitudes of $g$ and $\Psi_{nl}$. As we mentioned above, it is connected with the nonideality of the measurement. However, with the increase of the nonlinear phase shift $\Psi_{nl}$ the $K_1$ and $K_2$ curves as functions of $g$ become ‘more abrupt’. For example, for $\Psi_{nl} \approx 9.5$ and $g = \pi/6$ (when condition (41) still holds) the numerical magnitudes of the correlation coefficients are $K_1 \approx 0.99$ and $K_2 \approx 0.97$ (see figures 7(a) and (b)). Thus, the accuracy of the measurement of the $S_{in}$ Stokes parameter is rather high.

It should be pointed out that the QND apparatus including only a single linear system, cannot realize the QND measurement of the Stokes parameter $S_{in}$. Indeed, the conditions (35) and/or (43) cannot be satisfied without two necessary demands, first, the nonlinear medium has to be anisotropic over the cubic nonlinearity and second, the PS light should arrive at the input of the linear system. In particular, for the two limiting cases considered above we have that the correlation coefficients $K_2 \rightarrow 0$, $K_1 \approx 1$ if the value $\lambda_2 \rightarrow 0$ or $K_1 \rightarrow 0$, $K_2 \approx 1$ for $\lambda_1 \rightarrow 0$ under the condition that $\Psi_{nl} = 0$ (see figures 7(a) and (b)).

2.3. QND measurements of the $S_2$ and $S_3$ Stokes parameters

Let us start with the analysis of measurement of the Stokes parameter $S_3$. According to the general procedure of the QND measurement described above, the Stokes parameter $S_3$ can be measured by the probe parameter $S_1$. For this purpose the linear coupling (32b), (32c) between these values could be realized in a linear spatially inhomogeneous optical fibre. Therefore, we can offer a scheme for the QND measurement of the $S_3$ Stokes parameter by modifying the scheme of the $S_1$ measurement.

The corresponding set-up is shown in figure 8. The QND apparatus contains two linear systems $L_1$ and $L_2$ and an anisotropic medium of cubic nonlinearity placed between them.
Figure 8. Scheme of the QND measurement of the $S_3$ Stokes parameter: $S_j$ ($j = 1, 3$) are the values of the Stokes parameters at the input of the QND apparatus consisting of a nonlinear medium $NL$ and linear systems $L_1$ and $L_2$. The $S_j^{\text{out}}$ are the Stokes parameters at the output of the linear system $L_2$.

The scheme in figure 8 differs from the other one in figure 6 since there is the addition of a linear device $L_1$, which gives the opportunity to 'turn' the preliminary Stokes parameters.

Let us consider the transformations for the Stokes parameters carried out by the QND apparatus shown in figure 8. As a result of propagation of light through the linear system $L_1$ we have

\begin{align*}
S_{2b} &= S_{2c} \quad (44a) \\
S_{1b} &= \lambda'_1 S_{1c} - \lambda'_2 S_{3c} \quad (44b) \\
S_{3b} &= \lambda'_1 S_{3c} + \lambda'_2 S_{1c} \quad (44c)
\end{align*}

where $S_{jc}$ ($j = 1, 2, 3$) is the Stokes parameter operator at the input of the $L_1$ system and $S_{jb}$ is the operator at its output. The coefficients $\lambda'_1, \lambda'_2$ can be found from the calculations of the transformation of two orthogonally polarized modes in the linear system $L_1$. For the special values $\lambda'_1 = 0$ and $\lambda'_2 = -1$ we get

\begin{align*}
S_{2b} &= S_{2c} \quad (45a) \\
S_{1b} &= S_{3c} \quad (45b) \\
S_{3b} &= -S_{1c} \quad (45c)
\end{align*}

It is easy to see from (45b), (45c) that the linear medium $L_1$ replaces the $S_3$ Stokes parameter by another one, i.e. the $S_1$ parameter, but does not touch the parameter $S_2$. Thus the information about the Stokes parameter $S_{3c}$ could be obtained by the Stokes parameter $S_{1b}$ at the input of the nonlinear medium (see equation (45b) and figure 8) without any demolition of the parameter $S_2$.

The difference of the photon numbers for two orthogonal modes, which is exactly the Stokes parameter $S_{1b}$, is the conserved value of the wave interaction in the nonlinear medium (see the details in the appendix). At the input of linear system $L_2$ we have

\begin{equation}
S_{1b}^{\text{out}} = S_{1b} = S_{3c} \quad (46)
\end{equation}

In the ideal case of the $S_3$ QND measurement we must have that $S_{1b}^{\text{out}} = S_{1b}^{\text{in}} = S_{3c}$ and $S_{3b}^{\text{out}} = S_{3b}^{\text{in}} = S_{3c}$ after the transformation of the Stokes parameters by the linear system $L_2$. However, these relations cannot be satisfied simultaneously.

For characterization of the 'quality' of the analysed QND measurement of the $S_3$ Stokes parameter we introduce the correlation coefficients between the values of the Stokes parameters at the input and output of the QND apparatus (cf equations (27) and (28)):

\begin{equation}
G_1 = \frac{|\langle S_3 S_3^{\text{in}} \rangle + \langle S_3^{\text{out}} S_3^{\text{in}} \rangle - 2\langle S_3^{\text{in}} \rangle \langle S_3^{\text{out}} \rangle|^2}{4 \langle \Delta S_3^{\text{out}} \rangle^2} \quad (47a)
\end{equation}
\[ G_2 = \frac{|(S_{3c} S_3^{\text{out}}) + (S_3^{\text{out}} S_{3c}) - 2(S_{3c})(S_3^{\text{out}})|^2}{4((\Delta S_3^{\text{in}})^2)((\Delta S_3^{\text{out}})^2)} \]  

(47b)

where \( S_3^{\text{out}} \) is the output probe Stokes parameter and \( S_{3c} \) (\( S_{3c}^{\text{out}} \)) is the measured Stokes parameter at the input (output) of the QND apparatus. The correlation coefficient (47a) describes the ability of the scheme to avoid the degradation of the Stokes parameter \( S_{3c} \); the correlation coefficient (47b) is related to the accuracy with which the \( S_{3c} \) parameter value can be obtained by the \( S_3^{\text{out}} \) probe quantity detection.

Substituting (46) for (47) we obtain

\[ G_1 = K_2 = \frac{|(S_3^{\text{in}} S_3^{\text{out}}) + (S_3^{\text{out}} S_3^{\text{in}}) - 2(S_3^{\text{in}})(S_3^{\text{out}})|^2}{4((\Delta S_3^{\text{in}})^2)((\Delta S_3^{\text{out}})^2)} \]  

(48a)

\[ G_2 = K_1 = \frac{|(S_3^{\text{in}} S_3^{\text{out}}) + (S_3^{\text{out}} S_3^{\text{in}}) - 2(S_3^{\text{in}})(S_3^{\text{out}})|^2}{4((\Delta S_3^{\text{in}})^2)((\Delta S_3^{\text{out}})^2)} \]  

(48b)

The relations (48a) and (48b) show that in the case of the precise measurement of the \( S_{3c} \) parameter the coefficients \( G_1 \) and \( G_2 \) are in agreement with the coefficients \( K_2 \) and \( K_1 \) considered above. The analogy between the inequalities (35) and (43) results in the following conditions for the \( S_{3c} \) Stokes parameter QND measurement (we consider the case when \( G_1 \approx 1 \) and \( G_2 \rightarrow 1 \)):

\[ ((\Delta S_3^{\text{in}})^2) \ll g_0^2((\Delta S_1^{\text{in}})^2) = g_0^2(\Delta S_{3c}^2) \]  

(49a)

\[ r_{13}^2 \ll 1 \]  

(49b)

where the linear coefficient \( g_0 = 2\beta_0 z_0 \ll 1 \) (see equation (43a)): \( ((\Delta S_3^{\text{in}})^2) \) and \( ((\Delta S_1^{\text{in}})^2) \) are the variances of the Stokes parameters \( S_3^{\text{in}} \) and \( S_1^{\text{in}} \) at the input of the linear system \( L2 \), respectively, and \( (\Delta S_{3c}^2) \) is the variance of the \( S_3 \) measured quantity at the entrance to the QND apparatus (i.e. linear system \( L1 \)). Here we also used the expression (46) and the correlation coefficient \( r_{13} \) between \( S_1^{\text{in}} \) and \( S_3^{\text{in}} \) (see equation (14)).

It is important that the Stokes parameter \( S_3^{\text{in}} \) at the output of the medium \( NL \) (figure 8) does not contain any information about the \( S_{3c} \) measured value. Indeed, it follows from the expression (45c) that the \( S_{3b} \) Stokes parameter at the input of the anisotropic nonlinear medium corresponds to the primary difference of the photon numbers of the two orthogonally polarized modes, i.e. to the parameter \( S_{3c} \).

Thus, realization of the conditions (49) of the QND measurement or the \( S_{3c} \) Stokes parameter depends on the linear transformation coefficient \( g_0 \) in the system \( L2 \) and also on the redistribution of the fluctuations in an anisotropic medium with cubic nonlinearity (see equation (49a)). Such a QND measurement procedure of the \( S_{3c} \) Stokes parameter is separated from the destructive influence of quantum fluctuations of another \( S_{2c} \) Stokes parameter because the last one is preserved in the linear systems \( L1 \) and \( L2 \) (see equations (44a) and (45a)).

Finally, we shall briefly consider an opportunity for QND measurement of the \( S_2 \) Stokes parameter. Such a measurement becomes possible within the framework of the scheme offered above (figure 8), if we use the property of symmetry of the Stokes parameters \( S_j \) (\( j = 0, 1, 2, 3 \)) and also developed at their rotation.

In particular, we should place an additional linear system \( L3 \) at the input of the QND apparatus (figure 8), executing the transformation of the Stokes parameters \( S_j \) (\( j = 0, 1, 2, 3 \)) similar to (31).

Let us fix the initial difference of phases \( \Phi = \pi/2 + 2\pi m \) (\( m = 0, 1, 2, \ldots \)) and choose the linear coupling coefficient of the orthogonally polarized modes in the system.
$g = 2\pi k$ ($k = 1, 2, 3, \ldots$). Then the following relations for the Stokes parameters $S_j$ hold at the input of the QND apparatus:

\begin{align}
S_{0c} &= S_{0d} \\
S_{1c} &= S_{1d} \\
S_{2c} &= -S_{3d} \\
S_{3c} &= S_{2d}
\end{align}

(50a)\quad(50b)\quad(50c)\quad(50d)

where $S_{jd}$ are the Stokes parameter operators at the input of the linear system $L3$ and $S_{jc}$ are their values at the input of the QND apparatus (figure 8).

From the expressions (50c) and (50d) one can see that the Stokes parameters $S_2$ and $S_3$ replace one another at the input of the QND apparatus. Thus, when the QND apparatus is tuned to measure the quantity $S_{3c}$ (figure 8) it also has the ability to measure the Stokes parameter $S_{2d}$ without any destruction (see equation (50d)).

2.4. Phase-sensitive measurement

The precise measurement procedure described above could also be used to carry out phase-sensitive measurements in quantum optics.

In fact, we can rewrite the parameters $S_2$ and $S_3$ in the form of the phase operators introduced by Susskind and Glogower (in the quasi-classical approximation corresponding to the case when the photon numbers $(n_x, n_y) \gg 1$, cf [12]; here $n_x$ and $n_y$ are the photon number operators for the modes polarized along the $x$ and $y$ axes, respectively (see e.g. [15]):

\begin{align}
S_2 &\approx 2(n_x n_y)^{1/2} \cos \Phi_- \\
S_3 &\approx 2(n_x n_y)^{1/2} \sin \Phi_-
\end{align}

(51a)\quad(51b)

where $\cos \Phi_- = \cos(\Phi_y - \Phi_x)$ and $\sin \Phi_- = \sin(\Phi_y - \Phi_x)$ are the cosine and sine operators of the phase difference for polarized modes (a more accurate description could be given using the Pegg–Barnett formalism, cf [15]).

Thus, we can obtain information about the phase difference of two polarized modes by the QND measurement of the $S_2$ and $S_3$ parameters. A more detailed analysis (which will be published elsewhere, see [16]) shows that a photon number, i.e. amplitude-squeezed light (when $\langle \Delta n_{x,y}^2 \rangle \ll \langle n_{x,y} \rangle$) should be applied to the QND apparatus. In this case the quantum fluctuations of the modes’ amplitudes, see equation (51), are suppressed and so the phase difference of two modes is ‘purely’ measured (cf [17]).

3. Conclusion

In the present paper an opportunity to generate polarization-squeezed state (PS) light in a spatially inhomogeneous nonlinear medium with high efficiency of energy exchange between two linearly polarized modes is demonstrated. The expressions obtained for the variances of the Stokes parameter of light show the possibility of redistribution of quantum fluctuations between them. Optimum conditions for PS light formation depend on the parameters of the problem: the coupling coefficient of the waves $\beta$ and the nonlinear parameter $\psi_{\text{eff}}$. At the same time the magnitude of the ratio of the initial phases for two interacting modes (determining the character of energy exchange in a system) is very important.

The possible applications of the discussed polarization-squeezed states of light and also the analysis of experimental realization of the considered QND measurement are the subject
of a separate study. We will investigate the problems of quantum cryptography, quantum computing and the EPR paradox in cases where the polarization parameters of light are concerned. Results in this direction will be presented in a forthcoming paper.

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Appendix

Here we consider the method of preparation of the optical field for QND measurement of the $S_1^n$ Stokes parameter by the procedure described in [7]. Let us suppose that an optical field propagates through an anisotropic medium with a third-order nonlinearity (i.e. Kerr-like medium), see figure 6. The two orthogonally polarized modes, described by the operators $b_x$ and $b_y$, are obtained in the approximation $\gamma_1,2 \phi \approx 0$, i.e. $b_{x,y}^\dagger b_{x,y}$ and therefore conditions (36) are fulfilled.

As we have mentioned in section 2 it is necessary to satisfy the following two conditions for the QND measurement of the Stokes parameter $S_1^n$. Firstly, the $S_3$ parameter fluctuations must be suppressed, i.e. PS light is required. Secondly, the absence of correlations between the Stokes parameters is also necessary ($r_{13} = r_{12} = 0$).

Taking into account (14) and (A1) we obtain for the correlation coefficients $r_{13}$ and $r_{12}$:

$$r_{13} \approx \frac{2e^{-W}(n) \sin(u) \cos(\phi)}{\langle (\Delta S_1^n)^2 \rangle / 2\langle n \rangle}$$  \hspace{1cm} (A2)

$$r_{12} \approx -\frac{2e^{-W}(n) \sin(u) \sin(\phi)}{\langle (\Delta S_1^n)^2 \rangle / 2\langle n \rangle}$$  \hspace{1cm} (A3)

where the variance $\langle (\Delta S_1^n)^2 \rangle$ is given by the expression (see [7])

$$\langle (\Delta S_1^n)^2 \rangle \approx 2\langle n \rangle(1 + \langle n \rangle(4W \cos^2(\phi) - \Delta \gamma \sin(2\phi)))$$ \hspace{1cm} (A4)

Here we use $\langle n \rangle = \langle (b_{x,y}^\dagger b_{x,y}) \rangle = \langle (\Delta S_3^m)^2 \rangle, u = \gamma - 0.5(\gamma_1 + \gamma_2), W = 0.5\langle n \rangle(\gamma - \gamma_1)^2 + (\gamma - \gamma_2)^2, \phi = \psi + \Psi_{nl}, \Delta \gamma = \gamma_3 - \gamma_1, \Psi_{nl} = \langle n \rangle \Delta \gamma$ is the effective nonlinear phase shift, $\psi$ is the phase mode difference at the input of the medium. The expressions (A2) and (A3) are obtained in the approximation $\gamma_1,2, \gamma \ll 1$ and $\langle n \rangle \gamma_1^2, \langle n \rangle \gamma^2 \ll 1$.

It follows from (A2) and (A3) that $r_{13} = r_{12} = 0$ for the case where $u = 0$, i.e. $\gamma = (\gamma_1 + \gamma_2)/2$. In this case we can regulate the level of quantum fluctuations, $\langle (\Delta S_1^n)^2 \rangle$, by phase $\phi$ and $\psi$. For example, from (A4) for $\gamma = (\gamma_1 + \gamma_2)/2$ and the extremal phase value $\phi = -0.5 \arctan(2/\Psi_{nl})$ we find

$$\langle (\Delta S_1^n)^2 \rangle = 2\langle n \rangle(1 - 0.5\Psi_{nl)((4 + \Psi_{nl}^2)^{1/2} - \Psi_{nl}))$$ \hspace{1cm} (A5)

The expression (A5) shows the possibility of significant suppression of the Stokes parameter fluctuations, and so, the $S_3$ Stokes parameter variance may be lower than one for the coherent state. Thus, polarization-squeezed light is generated.
It seems essential to emphasize that the value of \( \langle (\Delta S_1^3)^2 \rangle \) in (A4) corresponds to the case of ideal squeezing when the uncertainty relation (13) is minimum and the correlation coefficient \( r_{23} = 0 \).

Thus, under certain conditions for the parameters \( \gamma_1, \gamma_2 \) and \( \gamma \) there is no correlation for the Stokes parameters i.e. \( r_{13} = r_{23} = r_{12} = 0 \), at the output of the anisotropic cubic-nonlinearity medium. At the same time ideal PS light is formed.

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